A NOTE ABOUT TWO PROPERTIES OF MATRIX RINGS

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ABSTRACT

We show that there exists for each $m \geqq 2$ a (non-commutative) integral domain R with a nilpotent matrix $C \in R_m$ whose order of nilpotency is greater than m, and any $A \in R_m$ with a right (or a left) inverse is invertible.

Following an example given in [2, p. 35] it is shown in [4] that a ring which satisfies the condition N_m: *If* $C \in R_m$ *is nilpotent then* $C^m = 0$, it also satisfies the condition I_m: *If* $A, B \in R_m$ *and* $AB = 1$ *then* $BA = 1$, *which is denoted by* III in [1]. The aim of this note is to show that ther exists an integral domain which satisfies I_m but does not satisfy N_m.

First we prove the following:

LEMMA 1. *Let* U *be a ring with* 1 *and* V *an ideal in* U. *Let* $a_i, b_i \in U$, $i = 0, 1, 2, \cdots$, *and define for* $r = 0, 1, 2, \cdots$

$$(1) \qquad c_r = \sum_{i=0}^{r} a_i b_{r-i}, \quad d_r = \sum_{i=0}^{r} b_i a_{r-i}.$$

If $a_0 b_0 = b_0 a_0 = 1$ *and* $c_r \in V$ *for* $r > 0$, *then* $d_r \in V$ *for* $r > 0$.

Proof. By induction on r. Assume $d_i \in V$ for $0 < i \leqq r$ and proceed to prove $d_{r+1} \in V$. For $r = 0$ the proof will show that $d_1 \in V$. Consider the following series in the ring of formal power series $U[[x]]$:

$$f = \sum_{i=0}^{\infty} a_i x^i, \; g = \sum_{i=0}^{\infty} b_i x^i, \; h = \sum_{i=0}^{\infty} c_i x^i, \; k = \sum_{i=0}^{\infty} d_i x^i.$$

By (1) it follows that $h = fg$ and $k = gf$, hence we have $fk = hf$ and the coefficient of x^{r+1} in both sides of this equation is

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$$\sum_{i=0}^{r+1} a_i d_{r+1-i} = \sum_{i=0}^{r+1} c_{r+1-i} a_i.$$

But $c_0 = a_0 b_0 = 1$ and $d_0 = b_0 a_0 = 1$, hence $a_{r+1} d_0 = a_{r+1} = c_0 a_{r+1}$, and we obtain $a_0 d_{r+1} = \sum_{i=0}^{r+1} c_{r+1-i} a_i - \sum_{i=1}^{r+1} a_i d_{r+1-i} \in V$ since $c_1, \cdots, c_{r+1}, d_1, \cdots, d_r \in V$. It follows that $d_{r+1} = b_0 a_0 d_{r+1} \in V$.

Now let us consider the ring $R = P/T$ constructed in [3] with $k \geqq m \geqq 2$ which is an integral domain. We have:

THEOREM 2. *R satisfies I_l for each $l \geqq 1$.*

Proof. Since $R_l \simeq P_l/T_l$ it suffices to show that if $A, B \in P_l$ and $AB - 1 \in T_l$ then $BA - 1 \in T_l$. We write A, B as sums of homogeneous matrices (= matrices all of whose entries are of the same degree or 0): $A = \sum_{i=0}^{\infty} A_i$, $B = \sum_{i=0}^{\infty} B_i$; A_i, B_i are homogeneous of degree i. Denote by C_r and D_r the homogeneous components of AB and BA respectively, $r = 0, 1, 2, \cdots$. Then we have $AB = \sum_{r=0}^{\infty} C_r$, $BA = \sum_{r=0}^{\infty} D_r$ and

$$C_r = \sum_{i=0}^{r} A_i B_{r-i}, \quad D_r = \sum_{i=0}^{r} B_i A_{r-i}.$$

Now T is a homogeneous ideal [3, lemma 4] and hence it follows that T_l is a homogeneous ideal in P_l. From $AB - 1 \in T_l$ it follows $C_0 - 1 \in T_l$ and $C_r \in T_l$ for $r = 1, 2, \cdots$. Since T_l does not contain non-zero homogeneous matrices of zero degree we have $C_0 = 1$ and hence $A_0 B_0 = 1$. But A_0, B_0 are matrices over a field, hence $D_0 = B_0 A_0 = 1$. Thus, we may apply the lemma for P_l, T_l, A_i, B_i, C_r, D_r replacing U, V, a_i, b_i, c_r, d_r respectively, and we obtain $D_r \in T_l$ for $r = 1, 2, \cdots$. Hence $BA - 1 = \sum_{r=1}^{\infty} D_r \in T_l$ and this proves the theorem.

Note that the same proof shows that the ring $\mathscr{R}$ constructed in [3] also satisfies I_l and the same is true for each ring which is a homomorphic image of a subring of a formal power series ring, in any number of noncommutative indeterminates over a field, with homogeneous kernel.

Now, we restrict the conditions N_m and I_m to integral domains and identify each condition with the class of rings in which it holds. Since N_m implies I_m and the integral domain in the theorem does not satisfy N_m [3, Th. 1] we obtain:

THEOREM 3. *For each $m \geqq 2$, $N_m \subsetneqq I_m$ and $\bigcap_{l=1}^{\infty} N_l \subsetneqq \bigcap_{l=1}^{\infty} I_l$.*

Note that since in an integral domain an element with a right inverse is invertible, we have $N_1 = I_1$. If we admit zero-divisors it is easy to obtain for each $m \geqq 1$

a ring which satisfies I_m and does not satisfy N_m. Indeed, let F be a field and take $R = F_n$ with $n > m$. Clearly in R_m a matrix with a right inverse is invertible but R and hence also R_m contain nilpotent elements whose order of nilpotency is $n > m$.

REFERENCES

1. P. M. Cohn, *Some remarks on the invariant basis property*, Topology **5** (1966), 215-228.

2. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, R. I., 1964.

3. A. A. Klein, *Rings nonembeddable in fields with multiplicative semigroups embeddable in groups*, J. Algebra **7** (1967), 100-125.

4. A. A. Klein, *Necessary conditions for embedding rings into fields*, Trans. Amer. Math. Soc. **137** (1969), 141-151.

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